

An elementary approach to simplexes in thin subsets of Euclidean space

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ABSTRACT. We prove that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 3$, is greater than $\min \left\{ \frac{dk+1}{k+1}, \frac{d+k}{2} \right\}$, then the $\binom{k+1}{2}$ -dimensional Lebesgue measure of $T_k(E)$, the set of congruence classes of k -dimensional simplexes with vertices in E , is positive. This improves the best bounds previously known, decreasing the $\frac{d+k+1}{2}$ threshold obtained in [6] to $\frac{d+k}{2}$ via a different and conceptually simpler method. We also give a simpler proof of the $d - \frac{d-1}{2d}$ threshold for d -dimensional simplexes obtained in [10, 9].

1. Introduction

The classical Falconer distance problem, introduced by Falconer in [8] (see also [14] for the background information) is to find the dimensional threshold $s_0 = s_0(d)$ such that if the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than s_0 , then the Lebesgue measure of $\Delta(E) := \{|x - y| : x, y \in E\}$ is positive. Here, and throughout, $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$, the usual Euclidean distance. The best result known in the general setting, due to Wolff [19] for $d = 2$ and Erdoğan [5] for $d \geq 3$, is the dimensional threshold $s_0 = \frac{d}{2} + \frac{1}{3}$. If the set E is assumed to be Ahlfors-David regular, then the dimensional threshold $s_0 = 1$ was recently nearly established for $d = 2$ by Orponen [15]. More precisely, he showed that if the Hausdorff dimension of an Ahlfors-David regular set in the plane is 1, then the upper Minkowski dimension of the distance set is also 1. An example due to Falconer [8] shows that $s_0 = 1$ is essentially sharp: if the Hausdorff dimension of a planar set is less than one, then the upper Minkowski dimension of the distance set can in general be less than 1.

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1.1. Congruence and similarity classes of simplexes. The distance problem can be viewed as a question about two-point configurations. A pair of points $x, y \in E$ is congruent to another pair $x', y' \in E$ iff $|x - y| = |x' - y'|$. This induces an equivalence relation \sim and we may view $\Delta(E)$ as $E \times E \setminus \sim$. This set can be naturally identified with the distance set $\Delta(E)$. In the same way, we may consider $(k+1)$ -point configurations $\{x^1, x^2, \dots, x^{k+1}\}$, $x^j \in E$, $k \leq d$, and say that $\{x^1, \dots, x^{k+1}\}$ is *congruent* to $\{y^1, \dots, y^{k+1}\}$ if there exists a translation $\tau \in \mathbb{R}^d$ and a rotation $g \in O_d(\mathbb{R})$ such that $y^j = \tau + gx^j$, $j = 1, 2, \dots, k+1$. The resulting equivalence relation allows us to define $T_k(E) := E \times E \times \dots \times E \setminus \sim$, which can be identified with a $\binom{k+1}{2}$ -tuple of distances $|x^i - x^j|$, $1 \leq i < j \leq k+1$. We can also consider $T_k(E)$ as the set of equivalence classes of k -dimensional simplexes determined by points of E , and can be viewed as a subset of $\mathbb{R}^{\binom{k+1}{2}}$ for the purpose of measuring its size in terms of Hausdorff dimension of non-vanishing Lebesgue measure.¹

The natural generalization of the Falconer distance problem in this context is to find a dimensional threshold s_0 such that if the Hausdorff dimension, $\dim_{\mathcal{H}}(E)$, of a compact set E is greater than $s_0 = s_0(k, d)$, then the $\binom{k+1}{2}$ -dimensional Lebesgue measure of $T_k(E)$ is positive: $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$. A variety of results have been obtained in this direction in recent years using everything from multi-linear theory to group actions. See, for example, [10, 6, 9]. To various extents those papers were preceded and motivated by finite field models worked out in [1, 2, 4, 12]. As a result of these efforts we know that for a compact set $E \subset \mathbb{R}^d$ with $d \geq 3$ and $2 \leq k \leq d$, if

$$(1.1) \quad \dim_{\mathcal{H}}(E) > \min \left\{ \frac{dk+1}{k+1}, \frac{d+k+1}{2} \right\},$$

then $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$. The purpose of the current paper is to improve (i.e., lower) the $\frac{d+k+1}{2}$ threshold obtained in [6] down to $\frac{d+k}{2}$, and introduce a new method in doing so.

Closely related to these questions are estimates for multilinear forms which, borrowing a term for analogous expressions in the discrete setting, we call *incidence bounds*. For $\rho \in C_0^\infty(\mathbb{R}^d)$ with $\rho \geq 0$, $\text{supp}(\rho) \subset \{|x| \leq 1\}$, $\rho \equiv 1$ on $\{|x| \leq \frac{1}{4}\}$ and $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$, form the approximate identity $\rho^\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$. For $t > 0$, let σ_t denote the surface measure on the sphere of radius t , and $\sigma_t^\epsilon = \sigma_t * \rho_\epsilon$. Finally, let μ be a Frostman measure supported on a set $E \subset \mathbb{R}^d$, $d \geq 2$ (see [14]). For positive

¹For these considerations, one can ignore the action of the permutation group on (x^1, \dots, x^{k+1}) , since this does not affect the non-vanishing of the $\binom{k+1}{2}$ -dimensional Lebesgue measure.

numbers $\{t_{ij}\}$, we ask whether an incidence bound,

$$(1.2) \quad |\Lambda_{k,d}^\epsilon(\mu)| := \left| \int \dots \int \prod_{1 \leq i < j \leq k+1} \sigma_{t_{ij}}^\epsilon(x^i - x^j) \prod_{l=1}^{k+1} d\mu(x^l) \right| \lesssim 1,$$

holds. (Here and throughout, $X \lesssim Y$ means that there exists $C > 0$ such that $X \leq CY$ independent of ϵ .)

Whenever (1.2) holds, it implies that the $\binom{k+1}{2}$ -dimensional Lebesgue measure of $T_k(E)$ (defined above) is positive (see [9]). But the uniform estimate in (1.2) is considerably stronger than the positivity of the Lebesgue measure. For example, in [10], the authors proved that, in the case $k = d = 2$, (1.2) holds if $\dim_{\mathcal{H}}(E) > \frac{7}{4}$, yielding not only the continuous Falconer-type configuration result but also a discrete result: If $A \subset \mathbb{R}^2$ is a finite homogeneous set with $|A| = N$, then the number of triples of points from A determining an equilateral triangle of fixed side length does not exceed $CN^{\frac{9}{7}}$, an improvement over the previously known $Cn^{\frac{4}{3}}$ bound (which is a consequence of the Szemerédi-Trotter incidence theorem). (For applications of continuous incidence bounds in geometric measure theory, see, e.g., [7]; for the definition of homogeneous set, see [17].)

1.2. Statement of results. The main results of this paper are the following.

THEOREM 1.1. *Let E be a compact subset of \mathbb{R}^d , $d \geq 3$, $2 \leq k \leq d$. Suppose that*

$$(1.3) \quad \dim_{\mathcal{H}}(E) > \min \left\{ \frac{dk+1}{k+1}, \frac{d+k}{2} \right\}.$$

Then $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$.

REMARK 1.2. The following observations clarify the role of the exponents in Theorem 1.1.

- The estimate (1.3) improves (i.e., lowers) by $\frac{1}{2}$ the best exponent previously known (proved in [6] and described in (1.1)).
- The sufficiency as a lower bound of the first term in the min was previously obtained in [11].
- For $k = d = 2$, the best lower bound known is $\dim_{\mathcal{H}}(E) > \frac{8}{5}$.

THEOREM 1.3. (Incidence Bound) *Let E be a compact subset of \mathbb{R}^d , $d \geq 2$, $2 \leq k \leq d$ and let μ be a Frostman measure on E . Then (1.2) holds if*

$$\dim_{\mathcal{H}}(E) > \min \left\{ \frac{d+k}{2}, d - \frac{d-1}{2k} \right\}.$$

REMARK 1.4. The sufficiency as a lower bound of the second term in the min was previously obtained in [9]. The case $d = k = 2$ was handled earlier in [10]. Note that $d - \frac{d-1}{2k} < \frac{d+k}{2}$ if and only if $k = d$.

REMARK 1.5. Let $\alpha_{k,d}$ denote the optimal exponent for the congruent d -dimensional simplex problem, i.e., $\alpha_{k,d}$ is the infimum of those α for which $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$ whenever $\dim_{\mathcal{H}}(E) > \alpha$. Easy examples show that $\alpha_{k,d} \geq \max\{k-1, \frac{d}{2}\}$. For non-trivial sharpness examples Burak Erdoğan and the second listed author obtained $\alpha_{2,2} \geq \frac{3}{2}$, while Jonathan DeWitt, Kevin Ford, Eli Goldstein, Steven J. Miller, Gwyneth Moreland, the fourth listed author and Steven Senger obtained $\alpha_{k,d} \geq \frac{d(k+1)}{d+2}$ which is only non-trivial in the range $\frac{d}{2} < k \leq d$. The first novel case in Theorem 1.1, where the new threshold $\frac{d+k}{2}$ beats the previously obtained threshold $\frac{dk+1}{k+1}$, is when $d = 5, k = 2$. There we obtain the threshold $\frac{7}{2}$, which we can only compare to the trivial sharpness example $\alpha_{2,5} \geq 3$. The first new case where $\frac{d+k}{2}$ beats out the old threshold and where we have $k > \frac{d}{2}$ (in order to have a non-trivial sharpness example) is for $d = 7, k = 4$. In this case Theorem 1.1 yields the threshold $\frac{11}{2}$, while the non-trivial sharpness example shows $\alpha_{4,7} \geq \frac{35}{9}$. A gap remains.

The following multi-linear estimate follows easily from the proof of Theorem 1.3.

THEOREM 1.6. *For $\epsilon > 0$, let $\sigma^\epsilon = \sigma_1 * \rho_\epsilon$, and define the multi-linear operator*

$$M_k^\epsilon(f_1, \dots, f_k)(x^{k+1}) = \int \dots \int \prod_{1 \leq i < j \leq k+1} \sigma^\epsilon(x^i - x^j) \prod_{l=1}^k f_l(x^l) d\mu(x^l).$$

Then if μ is a compactly supported Frostman measure on a set $E \subset \mathbb{R}^d$ with $\dim_{\mathcal{H}}(E) > \min\{\frac{d+k}{2}, d - \frac{d-1}{2k}\}$, then

$$M_k^\epsilon : L^k(\mu) \times \dots \times L^k(\mu) \rightarrow L^1(\mu),$$

with constants independent of ϵ .

2. Proof of Theorem 1.1

As mentioned in the introduction, the fact that $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$ if $\dim_{\mathcal{H}}(E) > \frac{dk+1}{k+1}$ was established in [11]. We are thus left with establishing the $\frac{d+k}{2}$ threshold.

Begin with the following observation proved in [9]. Heuristically, it says that if a given a configuration does not arise too often, there must be many different configurations. In the language of combinatorics, this is an incidence estimate.

LEMMA 2.1. Let $E \subset \mathbb{R}^d$, $d \geq 2$, $2 \leq k \leq d$, μ be a Frostman measure on E , and σ_t , σ_t^ϵ be as above. Let $t_{ij} > 0$ be arbitrary. Suppose that

$$(2.1) \quad |\Lambda_k(\mu)| := \left| \int \dots \int \prod_{1 \leq i < j \leq k+1} \sigma_{t_{ij}}^\epsilon(x^i - x^j) d\mu(x^1) \dots d\mu(x^{k+1}) \right| \lesssim 1$$

with constants independent of ϵ . Then $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$.

REMARK 2.2. Observe that $\Lambda_k(\mu)$ depends on ϵ and we shall obtain bounds independent of ϵ .

REMARK 2.3. This incidence estimate also establishes the first threshold in Theorem 1.3.

2.1. The case $k = 2$. We now bound (2.1) in the case $k = 2$ because it is slightly simpler and illustrates the method. For the sake of notational simplicity, we write out the case $t_{12} = t_{13} = t_{23} = 1$, but the argument works in the general case of general t_{ij} . Denote σ_1^ϵ by σ^ϵ . We have

$$\Lambda_2(\mu) = \int \int \int \sigma^\epsilon(x - y) \sigma^\epsilon(x - z) \sigma^\epsilon(y - z) d\mu(x) d\mu(y) d\mu(z).$$

Given complex numbers α, β, γ , define

$$\Lambda_2^{\alpha, \beta, \gamma}(\mu) = \int \int \int \sigma^{\epsilon, \alpha}(x - y) \sigma^{\epsilon, \beta}(x - z) \sigma^{\epsilon, \gamma}(y - z) d\mu(x) d\mu(y) d\mu(z),$$

where

$$\sigma^{\epsilon, z}(x) := \frac{2^{\frac{d-z}{2}}}{\Gamma(z/2)} (\sigma^\epsilon * |\cdot|^{-d+z})(x)$$

is initially defined for $\operatorname{Re}(z) > 0$ but then extended to the complex plane by analytic continuation. This follows the strategy introduced by the first two authors in [10].

THEOREM 2.4. Let E , μ and $\Lambda_2^{\alpha, \beta, \gamma}$ be as above.

Suppose that $\operatorname{Re}(\alpha) = 1 + \delta$, $\operatorname{Re}(\beta) = \operatorname{Re}(\gamma) = -\frac{1}{2} - \frac{\delta}{2}$ for some small $\delta > 0$. Then

$$(2.2) \quad |\Lambda_2^{\alpha, \beta, \gamma}(\mu)| \lesssim 1$$

with constants independent of ϵ provided that $\dim_{\mathcal{H}}(E) > \frac{d+2}{2}$.

COROLLARY 2.5. In view of Theorem 2.4 and the three lines lemma, (2.2) holds if $\alpha = \beta = \gamma = 0$ and $\dim_{\mathcal{H}}(E) > \frac{d+2}{2}$.

To prove (2.2) we shall need the following result due to the second named author, Krause, Sawyer, Taylor and Uriarte-Tuero [13].

THEOREM 2.6. Let μ, ν be compactly supported Borel measures satisfying $\mu(B(x, r)) \leq Cr^{s_\mu}$, $\nu(B(x, r)) \leq Cr^{s_\nu}$, and λ compactly supported Borel measure satisfying $|\widehat{\lambda}(\xi)| \leq C|\xi|^{-\eta}$. Define $T_{\lambda^\epsilon} f(x) = \int \lambda^\epsilon(x - y)f(y)d\mu(y)$. Let $s = \frac{s_\mu + s_\nu}{2}$, and suppose that $\eta > d - s$. Then

$$(2.3) \quad \|T_{\lambda^\epsilon} f\|_{L^2(\nu)} \leq C\|f\|_{L^2(\mu)}$$

with constant C independent of ϵ .

We include the proof of Theorem 2.6, for the sake of completeness, in Section 3 below. With Theorem 2.6 in tow, the proof of Theorem 2.4 proceeds as follows. Let $\operatorname{Re}(\alpha) = 1 + \frac{\delta}{2}$. Then

$$(2.4) \quad |\Lambda_2^{\alpha, \beta, \gamma}(\mu)| \leq C \int \int \int |\sigma^{\epsilon, \beta}|(x - z)|\sigma^{\epsilon, \gamma}|(y - z)d\mu(x)d\mu(y)d\mu(z).$$

Here we need the following basic calculation.

LEMMA 2.7. Let $\sigma^{\epsilon, \alpha}$ be as above. Then

$$|\sigma^{\epsilon, \alpha}(x - y)| \lesssim 1.$$

To prove this lemma first write

$$\sigma^{\epsilon, \alpha}(x - y) = \frac{2^{\frac{d-\alpha}{2}}}{\Gamma(\alpha/2)} \int \sigma^\epsilon(\tau)|x - y - \tau|^{-d+\alpha}d\tau,$$

and then decompose the integral in τ into annuli where $|x - y - \tau| \sim 2^{-j}$ and note that since we are in a compact setting then this happens for a set of j that are bounded below, say $-M \leq j$, where M only depends on the diameter of the set E . Finally conclude

$$\begin{aligned} |\sigma^{\epsilon, \alpha}(x - y)| &\lesssim \frac{2^{\frac{d-\operatorname{Re}(\alpha)}{2}}}{|\Gamma(\alpha/2)|} \sum_{j=-M}^{\infty} 2^{j(d-\operatorname{Re}(\alpha))} \int_{|x-y-\tau| \sim 2^{-j}} \sigma^\epsilon(\tau)d\tau \\ &\lesssim \frac{2^{\frac{d-\operatorname{Re}(\alpha)}{2}}}{|\Gamma(\alpha/2)|} \sum_{j=-M}^{\infty} 2^{j(d-\operatorname{Re}(\alpha))} 2^{-j(d-1)} \\ &\lesssim \frac{2^{\frac{d-\operatorname{Re}(\alpha)}{2}}}{|\Gamma(\alpha/2)|} \sum_{j=-M}^{\infty} \left(2^{\frac{\delta}{2}}\right)^{-j} \lesssim 1 \end{aligned}$$

This completes the proof of Lemma 2.7.

Applying Cauchy-Schwarz to (2.4) reduces matters to bounding

$$(2.5) \quad \int |\sigma^{\epsilon, \beta}| * \mu(x)^2 d\mu(x)$$

and the same expression with γ instead of β . It can be deduced from the classical stationary phase estimates (see, e.g., [18]) that

$$|\widehat{|\sigma^{\epsilon,\beta}|}(\xi)| \leq C|\xi|^{-\frac{d-1}{2}-\operatorname{Re}(\beta)}.$$

In view of this and the assumption that μ in the definition of $\Lambda_2^{\alpha,\beta,\gamma}(\mu)$ is a Frostman measure, we may apply Theorem 2.6 with $\mu = \nu$, s_μ slightly smaller than $\dim_{\mathcal{H}}(E)$ and $\eta = \frac{d-1}{2} - \operatorname{Re}(\beta) = \frac{d-2}{2} - \frac{\delta}{2}$. It follows that the expression in (2.5) is bounded if $\dim_{\mathcal{H}}(E) > d - \frac{d-2}{2}$, as claimed.

2.2. The general case. In the case $k = 2$ we reduced matters to the chain of length 2. In general, we shall reduce matters to chains of length k . Let

$$\Lambda_k^\alpha(\mu) = \int \dots \int \prod_{1 \leq i < j \leq k+1} \sigma_{t_{ij}}^{\epsilon, \alpha_{ij}}(x^i - x^j) d\mu(x^1) \dots d\mu(x^{k+1}).$$

Now set $\binom{k}{2}$ of the $\operatorname{Re}(\alpha_{ij})$ s equal to $1 + \delta \binom{k}{2}^{-1}$, where the α_{ij} s are chosen in such a way that what remains is a k -link chain. Choose the remaining α_{ij} s equal to $-\frac{k-1}{2} - \frac{\delta}{k}$. This is accomplished as follows. There are $(k+1)!$ ways to order $1, 2, \dots, k+1$. Each such ordering can be written as $(N(1), N(2), \dots, N(k+1))$, where N is a bijection on $\{1, 2, \dots, k+1\}$. Given such an ordering, we set $\operatorname{Re}(\alpha_{ij}) = -\frac{k-1}{2} - \frac{\delta}{k}$ if $(i, j) = (N(m), N(m+1))$ or $(N(m+1), N(m))$ (depending on whether or not $N(m) < N(m+1)$). For the remaining (i, j) , $1 \leq i < j \leq k+1$, set $\operatorname{Re}(\alpha_{ij}) = 1 + \delta \binom{k}{2}^{-1}$.

It follows from Lemma 2.7 that, up to a relabeling of vertices,

$$(2.6) \quad |\Lambda_k^\beta(\mu)| \leq \int \dots \int \prod_{j=1}^k |\sigma_{t_j}^{\epsilon, \beta_j}|(x^{j+1} - x^j) d\mu(x^1) \dots d\mu(x^{k+1}),$$

where

$$\operatorname{Re}(\beta_j) = -\frac{k-1}{2} - \frac{\delta}{k}.$$

We shall see using a modification of an argument in [3] that the right hand side of (2.6) is bounded if $\dim_{\mathcal{H}}(E) > \frac{d+k}{2}$. We shall need the following generalization of an upper bound from [3].

THEOREM 2.8. *Let $\lambda_1, \dots, \lambda_n$ be compactly supported Borel measures such that*

$$|\widehat{\lambda_j}(\xi)| \leq C|\xi|^{-\alpha} \text{ for some } \alpha > 0 \text{ and all } 1 \leq j \leq n.$$

Let $\lambda_j^\epsilon(x) = \lambda_j * \rho_\epsilon(x)$ where ρ_ϵ is as above. Let μ be a Frostman measure on a compact set of Hausdorff dimension s . Then

$$(2.7) \quad \left| \int \dots \int \prod_{j=1}^n \lambda_j^\epsilon(x^{j+1} - x^j) d\mu(x^1) \dots d\mu(x^{n+1}) \right| \leq C^{n+1}$$

independent of ϵ if $s > d - \alpha$. Here C is the constant obtained in Theorem 2.6.

Using the fact that

$$|\widehat{\sigma_{t_j}^{\epsilon, \beta_j}}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2} - \operatorname{Re}(\beta_j)}$$

where $\operatorname{Re}(\beta_j) = -\frac{k-1}{2} - \frac{\delta}{k}$ allows us to conclude, using Theorem 2.8, that the right hand side of (2.6) is bounded if

$$\dim_{\mathcal{H}} E > d - \left(\frac{d-1}{2} - \frac{k-1}{2} - \frac{\delta}{k} \right) = \frac{d+k}{2} + \frac{\delta}{k}$$

for any $\delta > 0$. Finally observe that the sum of the $\operatorname{Re}(\alpha_{ij})$ is 0 which shows that $\vec{0}$ is in the convex hull of all the points obtained by permuting the α_{ij} and this completes the proof of Lemma 2.1 via the three lines lemma.

To prove Theorem 2.8 we follow a similar strategy as in [3]. Let

$$T_k^\epsilon f(x) = (\lambda_k * (f\mu))(x),$$

$f_0^\epsilon(x) = 1$, and then recursively define $f_{k+1}^\epsilon = T_{k+1}^\epsilon f_k^\epsilon$, $k \geq 0$. With this notation we can write (2.7) as

$$\left| \int f_{n+1}^\epsilon(x^{n+1}) d\mu(x^{n+1}) \right| \leq C.$$

We start with the left hand side and apply the Cauchy-Schwarz inequality,

$$\left| \int f_{n+1}^\epsilon(x^{n+1}) d\mu(x^{n+1}) \right| \leq \|f_{n+1}^\epsilon\|_{L^2(\mu)},$$

where we use that μ is a probability measure so $\int d\mu(x^{n+1}) = 1$. Then the proof is completed with repeated use of Theorem 2.6:

$$\|f_{n+1}^\epsilon\|_{L^2(\mu)} \leq C \|f_n^\epsilon\|_{L^2(\mu)} \leq \dots \leq C^{n+1} \|f_0^\epsilon\|_{L^2(\mu)} = C^{n+1}$$

where $\mu = \nu$, with s_μ slightly smaller than $\dim_{\mathcal{H}}(E)$ and $\alpha > d - s$.

3. Proof of Theorem 2.6

It is enough to show that if $g \in L^2(\nu)$, then

$$(3.1) \quad | \langle T_{\lambda^\epsilon} f, g \nu \rangle | \leq C \|f\|_{L^2(\mu)} \cdot \|g\|_{L^2(\nu)},$$

where the constant C is independent of ϵ .

The left hand side of (3.1) equals

$$(3.2) \quad \int \lambda^\epsilon * (f\mu)(x) g(x) d\nu(x).$$

Indeed,

$$\lambda^\epsilon * (f\mu)(x) = \int e^{2\pi i x \cdot \xi} \widehat{\lambda}(\xi) \widehat{\rho}(\epsilon \xi) \widehat{f\mu}(\xi) d\xi$$

for every $x \in \mathbb{R}^d$ because the left hand side is a continuous $L^2(\mathbb{R}^d)$ function and

$$\widehat{\lambda}(\cdot) \widehat{\rho}(\epsilon \cdot) \widehat{f\mu}(\cdot) \in L^1 \cap L^2(\mathbb{R}^d).$$

It follows that (3.2) equals

$$\int \int e^{2\pi i x \cdot \xi} \widehat{\lambda}(\xi) \widehat{\rho}(\epsilon \xi) \widehat{f\mu}(\xi) d\xi g(x) d\nu(x).$$

Applying Fubini, we see that this expression equals

$$\begin{aligned} & \int \int e^{2\pi i x \cdot \xi} g(x) d\nu(x) \widehat{\lambda}(\xi) \widehat{\rho}(\epsilon \xi) \widehat{f\mu}(\xi) d\xi \\ &= \int \widehat{\lambda}(\xi) \widehat{\rho}(\epsilon \xi) \widehat{f\mu}(\xi) \widehat{g\nu}(\xi) d\xi. \end{aligned}$$

The modulus of this expression is bounded by an ϵ -independent multiple of

$$\int |\xi|^{-\alpha} |\widehat{f\mu}(\xi)| \cdot |\widehat{g\nu}(\xi)| d\xi.$$

By Cauchy-Schwarz, this expression is bounded by

$$(3.3) \quad \left(\int |\widehat{f\mu}(\xi)|^2 |\xi|^{-\alpha_\mu} d\xi \right)^{\frac{1}{2}} \cdot \left(\int |\widehat{g\nu}(\xi)|^2 |\xi|^{-\alpha_\nu} d\xi \right)^{\frac{1}{2}} =: \sqrt{A} \cdot \sqrt{B},$$

where $\alpha_\mu, \alpha_\nu > 0$ and $\frac{\alpha_\mu + \alpha_\nu}{2} = \alpha$.

LEMMA 3.1. *With the notation above, we have*

$$A \leq C \|f\|_{L^2(\mu)}^2 \text{ and } B \leq C \|g\|_{L^2(\nu)}^2$$

if

$$(3.4) \quad \alpha_\mu > d - s_\mu \text{ and } \alpha_\nu > d - s_\nu.$$

We give a direct argument in the style of the proof of Lemma 7.4 in [20]. It is enough to prove that the estimate for A follows from the condition on α_μ, s_μ , since the estimate for B follows from the same statement applied to α_ν, s_ν . By Proposition 8.5 in [20],

$$(3.5) \quad A = \int \int f(x)f(y)|x-y|^{-d+\alpha_\mu} d\mu(x)d\mu(y) = \langle f, Uf \rangle,$$

where

$$Uf(x) = \int |x-y|^{-d+\alpha_\mu} f(y) d\mu(y).$$

Observe that

$$\begin{aligned} \int |x-y|^{-d+\alpha_\mu} d\mu(y) &= \int |x-y|^{-d+\alpha_\mu} d\mu(x) \\ &\leq C \sum_{j>0} 2^{j(d-\alpha_\mu)} \int_{2^{-j} \leq |x-y| \leq 2^{-j+1}} d\mu(y) \\ &\leq C' \sum_{j>0} 2^{j(d-\alpha_\mu-s_\mu)} \leq C'' \text{ if } \alpha_\mu > d - s_\mu. \end{aligned}$$

It follows by Schur's test (see Lemma 7.5 in [20] and the original argument in [16]) that

$$\|Uf\|_{L^2(\mu)} \leq C'' \|f\|_{L^2(\mu)}$$

and we are done in view of (3.5) and Cauchy-Schwarz.

4. Proof of the second estimate in Theorem 1.3

As mentioned in the introduction, the fact that $\mathcal{L}^{\binom{k+1}{2}}(T_k(E)) > 0$ if $\dim_{\mathcal{H}}(E) > d - \frac{d-1}{2k}$ was established in [9]. We give a simpler, more transparent proof below.

As we noted in Remark 1.4, we only need to deal with the case $k = d$, since for $k \leq d-1$, the known $\alpha_{k,d} \leq \frac{d+k}{2}$ is the minimum of the two quantities in Theorem 1.3. We prove the estimate in the case $t_{ij} = 1$ as the proof of the general case is essentially the same. Let μ be a Frostman measure on E and define $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{d+1})$, where $\alpha_j \in \mathbb{C}$. Define $\Lambda^\delta(\alpha)$ by the integral

$$\int \int \sigma^\epsilon(x^1 - x^2) d\mu_{\alpha_1}^\delta(x^1) d\mu_{\alpha_2}^\delta(x^2) \int \dots \int \prod_{1 \leq i < j \leq d+1; (i,j) \neq (1,2)} \sigma^\epsilon(x^i - x^j) \prod_{l=3}^{d+1} d\mu_{\alpha_l}^\delta(x^l),$$

where

$$\mu^\delta(x) = \mu * \rho_\delta(x),$$

with ρ a smooth cut-off function, $\int \rho = 1$, $\rho_\delta(x) = \delta^{-d} \rho(x/\delta)$, and

$$(4.1) \quad \mu_z^\delta(x) := \frac{2^{\frac{d-z}{2}}}{\Gamma(z/2)} (\mu^\delta * |\cdot|^{-d+z})(x).$$

We shall prove that when $Re(\alpha_j)$, $j = 3, \dots, d+1$, equals $\frac{d-1}{2d} - \frac{\epsilon}{d-1}$, then

$$(4.2) \quad \left| \int \dots \int \prod_{1 \leq i < j \leq d+1; (i,j) \neq (1,2)} \sigma^\epsilon(x^i - x^j) \prod_{l=3}^{d+1} d\mu_{\alpha_l}^\delta(x^l) \right| \leq C$$

provided that $\dim_{\mathcal{H}}(E) > d - \frac{d-1}{2d} + \frac{\epsilon}{d-1}$. This will follow from two observations. First, we are going to show that

$$(4.3) \quad |\mu_z^\delta(x)| \leq C$$

provided that $\dim_{\mathcal{H}}(E) > d - Re(z)$.

We will then show that

$$(4.4) \quad \left| \int \dots \int \prod_{1 \leq i < j \leq d+1; (i,j) \neq (1,2)} \sigma^\epsilon(x^i - x^j) dx_3 \dots dx_{d+1} \right| \leq C$$

using elementary geometric considerations. Combining (4.4) and (4.3) yields (4.2).

We shall then prove that if $Re(\alpha_1) = Re(\alpha_2) = -\frac{(d-1)^2}{4d} + \frac{\epsilon}{2}$, then

$$(4.5) \quad \int \int \sigma^\epsilon(x^1 - x^2) d|\mu_{\alpha_1}|(x^1) d|\mu_{\alpha_2}|(x^2) \leq C$$

under the same constraint on $\dim_{\mathcal{H}}(E)$ as above.

Combining (4.5) and (4.4) will show that

$$|\Lambda^\delta(\alpha)| \leq C$$

if $Re(\alpha_1) = Re(\alpha_2) = -\frac{(d-1)^2}{4d} + \frac{\epsilon}{2}$ and $Re(\alpha_j) = \frac{d-1}{2d} - \frac{\epsilon}{d-1}$, $j = 3, \dots, d+1$. Interchanging the role of variables, we obtain $\binom{d+1}{2}$ relations of this type and since the real parts add up to 0 we have $\vec{0}$ in the convex hull of all these relations and thus the result easily follows by analytic interpolation.

We now establishes estimates (4.3), (4.4), (4.5).

4.1. Proof of the estimate (4.3). We have

$$\begin{aligned}
|\mu_z^\delta(x)| &= \left| \frac{2^{\frac{d-z}{2}}}{\Gamma(z/2)} \right| \int |x-y|^{-d+z} d\mu_\delta(y) \\
&= \left| \frac{2^{\frac{d-z}{2}}}{\Gamma(z/2)} \right| \sum_j \int_{2^{-j-1} < |x-y| \leq 2^{-j}} |x-y|^{-d+z} d\mu^\delta(y) \\
&\leq C \left| \frac{2^{\frac{d-z}{2}}}{\Gamma(z/2)} \right| \sum_j 2^{j(d-\operatorname{Re}(z))} \int_{2^{-j-1} < |x-y| \leq 2^{-j}} d\mu^\delta(y) \\
&\leq C'_\alpha(z) \sum_j 2^{j(d-\operatorname{Re}(z)-\alpha)}
\end{aligned}$$

for any $\alpha < \dim_{\mathcal{H}}(E)$ since μ is a Frostman measure. The geometric series converges if $\operatorname{Re}(z) > d - \dim_{\mathcal{H}}(E)$, as claimed. This completes the proof of the estimate (4.3).

4.2. Proof of the estimate (4.4). This is a volume packing estimate. We have a transverse intersection of ϵ neighborhoods of $\binom{d+1}{2} - 1$ surfaces in $\mathbb{R}^{d(d-2)}$ multiplied by $\epsilon^{-\binom{d+1}{2}+1}$. This results in an ϵ neighborhood of an $d(d-2) - \binom{d+1}{2} + 1$ dimensional surface, multiplied by $\epsilon^{-\binom{d+1}{2}+1}$. The resulting weighted volume is ≈ 1 and the proof is complete.

4.3. Proof of the estimate (4.5). This result follows from the following general observation combined with (4.1).

LEMMA 4.1. *Let $d\mu_A, d\mu_B$ denote compactly supported Borel measures and s_A, s_B real numbers such that*

$$\max \left\{ \int |\widehat{\mu}_A(\xi)|^2 |\xi|^{-d+s_A} d\xi, \int |\widehat{\mu}_B(\xi)|^2 |\xi|^{-d+s_B} d\xi \right\} \leq C.$$

Let $\lambda \in L^1(\mathbb{R}^d)$ such that

$$|\widehat{\lambda}(\xi)| \leq C|\xi|^{-\gamma} \text{ for some } \gamma > 0.$$

Then

$$\int \int \lambda(x-y) d\mu_A(x) d\mu_B(y) \leq C' \text{ if } \frac{s_A + s_B}{2} > d - \alpha.$$

The proof is very similar to that of Theorem 2.6. Choose α_A, α_B such that $\frac{\alpha_A + \alpha_B}{2} = \alpha$ and write

$$\int \int \lambda(x-y) d\mu_A(x) d\mu_B(y) = \int \widehat{\mu}_A(\xi) \overline{\widehat{\mu}_B(\xi)} \widehat{\lambda}(\xi) d\xi$$

$$\begin{aligned}
&\leq \int |\widehat{\mu}_A(\xi)| |\widehat{\mu}_B(\xi)| |\xi|^{-\alpha} d\xi = \int |\widehat{\mu}_A(\xi)| |\xi|^{-\frac{\alpha_A}{2}} |\widehat{\mu}_B(\xi)| |\xi|^{-\frac{\alpha_B}{2}} d\xi \\
&\leq \left(\int |\widehat{\mu}_A(\xi)|^2 |\xi|^{-\alpha_A} d\xi \right)^{\frac{1}{2}} \cdot \left(\int |\widehat{\mu}_B(\xi)|^2 |\xi|^{-\alpha_B} d\xi \right)^{\frac{1}{2}} = I \cdot II.
\end{aligned}$$

By assumption, I is bounded if $\alpha_A \geq d - s_A$ and II is bounded if $\alpha_B \geq d - s_B$. This completes the proof of the lemma and hence the proof of the second estimate in Theorem 1.3.

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